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On List-Coloring and the Sum List Chromatic Number of Graphs

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Contents

Abstract	iv
1 Preliminaries	1
1.1 Standard Definitions	1
2 The List-Chromatic Number of a Graph	2
2.1 The Chromatic Number of a Graph and Brook's Theorem	2
2.2 Definition of List-Chromatic Number	4
2.3 List-Extension of Brooks' Theorem	5
2.4 Asymptotic Bounds for $\chi_l(G)$	8
2.5 The List-Chromatic Number of Planar Graphs	11
2.6 2-Choosability of Graphs	17
3 The Sum Choice Number of a Graph	23
3.1 Definition and First Results	23
3.2 Two Important Lemmas	25
3.3 Gathering the Tools	26
3.4 A Warm Up Before Coloring the Petersen Graph	28
3.5 Sum-Coloring the Petersen Graph	29
4 Concluding Remarks and Future Research	34
Bibliography	36

Abstract

ON LIST-COLORING AND THE SUM LIST CHROMATIC NUMBER OF GRAPHS

By Coleman David Hall, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

Virginia Commonwealth University, 2011.

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This thesis explores several of the major results in list-coloring in an expository fashion. As a specialization of list coloring, the sum list chromatic number is explored in detail. Ultimately, the thesis is designed to motivate the discussion of coloring problems and, hopefully, interest the reader in the branch of coloring problems in graph theory.

Preliminaries

1.1 Standard Definitions

Let the graph G be written in the form $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ denote the vertex set and edge set, respectively. In this paper, we assume all graphs to be undirected with no loops and no multiple edges. Let $\alpha(G)$ and $\omega(G)$ denote the *independence number* and *clique number*, respectively. A *proper vertex coloring* is a mapping $f : V \rightarrow \mathbb{N}$ such that $f(v) \neq f(v')$ for all adjacent vertices $v, v' \in V$ with $vv' \in E$. Throughout this paper, proper coloring and coloring will be used interchangeably. The *neighborhood* of a vertex v is the set of all vertices adjacent to v and is denoted $N(v)$. Let $\delta(v)$ and $\Delta(G)$ denote the *minimum degree* and *maximum degree* of all vertices of G . Any other definitions will be defined when necessary.

The focus of this exposition is restricted to vertex colorings. Extensive research has been and is currently conducted on edge colorings. For a survey of results in edge colorings, the reader should refer to [18]. For an excellent exposition on the chromatic number by Jensen and Toft, the reader should refer to [15] and [16]. N. Alon, R. Diestel, D. West, and Zs. Tuza develop list coloring extensively in [2], [7], [21], and [18], respectively. And for a survey of the sum list chromatic number by B. Heinold, the best sources are [11] and [12].

The List-Chromatic Number of a Graph

2.1 The Chromatic Number of a Graph and Brook's Theorem

A *vertex coloring* of a graph G is a mapping $f : V \rightarrow S$ where S denotes a set of colors, and the set of vertices of the same color form a *color class*. If $|S| = k$, then f is called a *k-coloring*, and we often take $S = \{1, 2, \dots, k\}$ to be the k -colors of S . A k -coloring is called *proper* if all adjacent vertices receive distinct colors. A graph is called *k-colorable* if it has a proper k -coloring. The *chromatic number* of a graph G , denoted $\chi(G)$, is the least k such that G is k -colorable. Such a coloring will always exist, for we can assign all vertices distinct colors, to yield a $|V|$ -coloring. An *optimal coloring* of G is a $\chi(G)$ -coloring. If $\chi(G) = k$, then G is called *k-chromatic*.

Since each color class forms an independent set, we can easily see that $\chi(G) \geq \frac{|V|}{\alpha(G)}$. To prove upper bounds for the chromatic numbers of graphs, we construct algorithms that produce proper colorings. The most common algorithm used is the *greedy coloring algorithm*. Order the vertices of V : v_1, v_2, \dots, v_n . A greedy coloring of V relative to the vertex ordering of V is obtained by coloring each v_i , $1 \leq i \leq n$, with the smallest indexed color of S that is not already assigned to any lower indexed vertex adjacent to v_i . The greedy coloring algorithm gives us an important upper bound. If we greedily color the vertices of a graph G , then each vertex of G has at most $\Delta(G)$ lower indexed neighbors. If all these lower indexed neighbors are adjacent to each other, then the greedy algorithm will yield, at worst, $\Delta(G) + 1$ colors. So for any graph G , $\chi(G) \leq \Delta(G) + 1$.

For the complete graph, K_n , and odd cycles, C_{odd} , the bound is optimal. Since every

vertex is adjacent to every other vertex in K_n , we must use at least n colors. Since $\Delta(G) = n - 1$, by our upper bound, $\chi(K_n) = n$. It follows that $\chi(G) \geq \omega(G)$.

Although the greedy coloring algorithm is simple to implement, one must strategically choose the vertex ordering. A vertex ordering that yields a proper coloring will always exist; however, if we are given a graph of order n , then there are $n!$ possible vertex orderings. Finding the ordering we desire can be a difficult problem. One of the first results on the chromatic number that successfully implements the greedy coloring algorithm is Brooks' Theorem. Lovasz (1975) is credited with this simplified proof of Brooks' Theorem. His proof creates a vertex ordering by building a tree from a root vertex. It also uses the fact that if a graph G is not complete, then there exists three vertices u, v , and w such that $uv \in E(G)$, $vw \in E(G)$, but $uw \notin E(G)$.

THEOREM 2.1. (Brooks' Theorem [Lovasz 1976]) Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ if G is neither complete nor an odd cycle.

Proof. If $\Delta(G) \leq 1$, then G is complete, and the bound holds. If $\Delta(G) = 2$, then G is either bipartite or an odd cycle; thus, in both situations, the bound holds. So assume $\Delta(G) \geq 3$.

First, assume G is not Δ -regular. Let v_n be a vertex of degree less than Δ . Let T be a spanning tree rooted at v_n . Assign indices in order of nonincreasing distance from v_n in T . Then we have constructed a vertex ordering v_1, v_2, \dots, v_n such that every vertex has a higher indexed neighbor except v_n . Greedily color the vertices, and we use at most Δ colors.

Now instead, assume that G is Δ -regular. If G has a cut vertex, call it v_c , then apply the above coloring procedure to each component of $G - v_c$. By permuting the colors in each component, we can make the same color appear on two neighbors of v_c . Now color v_c with some available color. This yields a proper Δ coloring of G .

So assume instead that G is 2-connected. Since G is not complete, there exists a vertex v_n such that two vertices, v_1 and v_2 , adjacent to v_n share no common edge. Consider

$G - \{v_1, v_2\}$. Let v_n be the root vertex of a spanning tree of $G - \{v_1, v_2\}$. Use the same vertex ordering as above, and greedily color the vertices. v_1 and v_2 will receive color 1, and when coloring v_n , at most $\Delta - 1$ colors will be used on its neighbors. Thus, the greedy coloring algorithm uses at most Δ colors. \square

The majority of our exposition deals with results in list-coloring; however, we still wish to implement the greedy coloring algorithm for list coloring problems. Brooks' Theorem for the chromatic number of a graph is an excellent example of constructing a vertex ordering and applying the algorithm, but a more relevant question arises. Does this same theorem hold for list-coloring? One would hope that analogous results and proof techniques would also work for list-coloring. When we begin with a vertex coloring theorem and make an analogous result for list coloring, one of three things will happen. The proof for the chromatic number works with minor adjustments, the proof is too weak to show the result, or the theorem is false!

2.2 Definition of List-Chromatic Number

Let L_i denote the list, which is a set of admissible colors, associated with each $v_i \in V$ for some graph G . Let $\mathbb{L} := L_1 \cup L_2 \cup \dots \cup L_n$ denote the union of all lists. Let $\mathcal{L} := (L_1, L_2, \dots, L_n)$ denote the ordered n -tuple of lists. A mapping $f : V \rightarrow \mathbb{L}$ is a *vertex list-coloring* if f is a proper coloring and $f(v_i) \in L_i$ holds for all $1 \leq i \leq n$. If $|L_i| = k$ for all i , then \mathbb{L} is called a *k-assignment*. The *list-chromatic number*, also called *choice number*, of a graph G , denoted $\chi_l(G)$, is the smallest k such that every k -assignment \mathbb{L} admits a list coloring. If $\chi_l(G) \leq k$, then G is said to be *k-choosable*. We also say that given some list assignment L , the graph is *L-choosable*.

List Coloring is a generalization of vertex coloring that was introduced independently by Erdos, Rubin, and Taylor [8] and Vizing [19]. Because we restrict the colors allowed at

each vertex, the proofs require more strategy for the algorithm constructions. If we choose all lists to be identical, say $L_i = \{1, 2, \dots, k\}$ for all i , then by definition of the chromatic number, $\chi(G)$ cannot exceed the list-chromatic number $\chi_l(G)$.

In section 2.1, we showed that $\chi(G) \leq \Delta(G) + 1$. Here we prove an analogous result for list coloring. The same bound for $\chi(G)$ stated above will also hold for list coloring, and it is referred to as the Szekeres-Wilf degeneracy bound. Define a *k-degenerate graph* to be a graph in which every subgraph of G has a vertex of degree at most k . Restated, some vertex in the subgraph will be incident with k or fewer of G 's edges. Formally stated, every k -degenerate graph is $(k + 1)$ -choosable, and moreover, $\chi_l(G) \leq \Delta(G) + 1$. To establish the bound, choose a vertex in $V(G)$ that has k neighbors. Since we are assuming G is k -degenerate, we have at most k neighbors. Iteratively continue in this fashion by picking the smallest degree vertices in nonincreasing order. We have constructed a vertex sequence. Reverse the order of the sequence, and greedily color the vertices. Since each vertex v is adjacent to at most k earlier neighbors, we can color v from its list.

By being so restrictive for the list sizes, we open a huge branch of questions. If we choose all lists to be identical, then we reduce the problem to a $\chi(G)$ -coloring problem. What if we restrict the cardinality of our lists to be greater than or equal to the degree of each vertex? Will Brooks' Theorem hold for list-coloring? Several choosability theorems are needed to show the result is true. These theorems are crucial for any individual studying list coloring, since they provide valuable techniques for showing other results.

2.3 List-Extension of Brooks' Theorem

We say a graph is *degree choosable* if it is L -choosable whenever $|L(v)| \geq d(v)$ for each vertex v . When we are working with graphs that have a considerable number of vertices and

edges, it is often easier to look at subgraphs. For list coloring and degree-choosability, if we have a connected graph that has a degree choosable subgraph, then the original graph is also degree choosable. To establish this fact we need the following lemma from [19].

LEMMA 2.2. Given a connected graph G , let L be a list assignment such that $|L(v)| \geq d(v)$ for all $v \in V(G)$. Then:

1. If $|L(y)| > d(y)$ for some vertex y , then G is L -colorable.
2. If G is 2-connected and the lists are not all identical, then G is L -colorable.

Proof. **(1.)** Color the vertices in order of nonincreasing distance to vertex y . Then every vertex has a neighbor that comes later in the ordering, except for y . When we color the vertex v directly before y in the sequence, we have colored fewer than $d(v)$ neighbors. So we have a color for v . Since $|L(y)| > d(y)$ by assumption and since we have colored at most $d(y)$ neighbors, we still have one color left to color y . This gives a proper coloring.

(2.) Since we are assuming the lists are not identical, then there are two adjacent vertices with distinct lists. So choose $v \in V(G)$ and $v^* \in N(v)$, and choose $c \in L(v) - L(v^*)$. We define new lists L' on $V(G - v)$ by $L'(u) = L(u)$ if $u \notin N(v)$ and $L'(u) = L(u) - c$ if $u \in N(v)$. Thus, $|L'(u)| \geq d_{G-v}(u)$ for all $u \in V(G - v)$ and $L'(v^*) > d_{G-v}(v^*)$. Apply part **(a)**. This yields an L' -coloring of $G - v$, and this extends to an L -coloring of G by using color c on v . □

LEMMA 2.3. If a connected graph G has a degree-choosable induced subgraph H , then G is degree-choosable.

Proof. Let L be a list assignment for G with $|L(v)| \geq d(v)$ for all v . In the graph $G - V(H)$, using the same lists as in G , every component has a vertex with $d(v) < |L(v)|$ since G is connected. By Lemma 2.2, $G - V(H)$ has a L -coloring f .

Now, form a list assignment L' for H . For each $v \in V(H)$ delete from $L(v)$ the colors

used by f on neighbors of v . Since $|L(v)| \geq d_G(v)$, we have $|L'(v)| \geq d_H(v)$ for $v \in V(H)$. Since H is degree choosable, H has an L' -coloring f' . Combining f and f' will yield a proper L -coloring of G . \square

In 1979, Erdős, Rubin, and Taylor showed that every 2-connected graph that is neither a complete graph nor an odd cycle has an even cycle with at most one chord [8]. The basic idea of the proof is finding the largest clique in the graph G and choosing shortest paths to a vertex that is not in the clique. A case analysis shows that depending on the parity of the length of the path we will get an even cycle with at most one chord. The only special case analysis required is if G is triangle free. Rather than explore the lengthy details of the proof, the result is stated as a lemma and will be used to extend Brooks' Theorem on degree-choosability to list coloring.

LEMMA 2.4. (Erdős-Rubin-Taylor [1979]) Every 2-connected graph G that is not a complete graph or an odd cycle has an even cycle with at most one chord.

In section 2.1, we proved that $\chi(K_n) = n$ and $\chi(C_{\text{odd}}) = 3$. Because of this, complete graphs and odd cycles are not degree-choosable. Define a *block* of a graph G to be a maximal 2-connected subgraph of G .

THEOREM 2.5. A graph G fails to be degree choosable if and only if every block is a complete graph or an odd cycle.

Proof. By Lemmas 2.3 and 2.4, we need only consider an even cycle with at most one chord. Let G be a graph such that every block is an even cycle with at most one chord. Call the even cycle with one chord H . Let L be the list assignment with $|L(v)| \geq d_H(v)$. Since H is 2-connected, by Lemma 2.2, it is L -colorable when all lists are not identical. If all the lists are identical, an even cycle is 2-colorable, and an even cycle with one chord is 3-colorable. \square

THEOREM 2.6. (List Extension of Brooks' Theorem, [21]) If G is a connected graph that is neither a complete graph nor an odd cycle, then $\chi_l(G) \leq \Delta(G)$.

Proof. (Contrapositive) Suppose that $\chi_l(G) > \Delta(G)$. The Szekeres-Wilf bound implies G is not $(\Delta(G) - 1)$ -degenerate. Hence, G has a subgraph H with $\delta(H) \geq \Delta(G)$. Thus, H is $\Delta(G)$ -regular. Since G is connected and vertices with degree $\Delta(G)$ in H have no neighbors outside H , we have $H = G$.

Theorem 2.5 implies that every block of G is a complete graph or an odd cycle. Since G is regular, G must have only one block since the cut-vertex in a leaf block would have higher degree than the other vertices. Thus, G is a complete graph or an odd cycle. \square

We have established our first upper bound for the list-chromatic number. So, $\chi(G) \leq \chi_l(G)$, but can we bound $\chi_l(G)$ in terms of $\chi(G)$?

2.4 Asymptotic Bounds for $\chi_l(G)$

We cannot upper bound $\chi_l(G)$ in terms of $\chi(G)$ in general. Erdős, Rubin, and Taylor showed that there exist bipartite graphs with arbitrarily large list chromatic number.

THEOREM 2.7. (Erdős, Rubin, and Taylor [1979]). If $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not k -choosable.

Proof. In both partite sets, let the lists be all possible k -sets of $\{1, 2, \dots, 2k-1\}$. In one partite set, P , we must use at least k distinct colors. For if not, we would have a vertex in P with no color chosen. Since we choose k colors from P , some vertex in the other partite set has this k set as its list. Hence, no proper coloring will exist. \square

An easy example is the case where $k = 2$ and $m = 3$. Here is the list assignment for $K_{3,3}$.

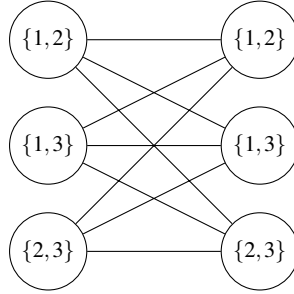


Figure 2.1: Example of $k = 2$.

Say we choose 1 and 2 as the k colors used on the left partite set. Then the list $\{1,2\}$ on the right will receive no color. The same is true for any other choices we make on the left. The idea, which is frequently used when constructing list counterexamples, is to make lists that will always force one vertex to be uncolored no matter what initial choices we pick. This idea will be used in the 2-choosability characterization of graphs later. For a more detailed example of $k = 3$ and $m = 10$, see [8].

In [10], G. Hajos built an algorithm based on three elementary operations and the repeated use of these three operations on K_{k+1} to generate all graphs of chromatic number greater than k . The three operations are as follows:

1. Add a new vertex or a new edge.
2. Let G_1, G_2 be vertex disjoint, and let $x_i y_i$ be an edge in G_i , $i = 1, 2$. Identify x_1 and x_2 , join y_1 with y_2 by a new edge, and delete the edges $x_1 y_1$ and $x_2 y_2$.

3. If G has an uncolorable list assignment \mathcal{L} such that $|L_i| \geq k$ for all $1 \leq i \leq n$ and two nonadjacent vertices $v_{i'}, v_{i''}$ have the same list in \mathcal{L} , then identify $v_{i'}$ with $v_{i''}$.

These three operations can be used to generate every non- k -choosable graph from any non- k -choosable complete bipartite graph. This result was proved by S. Gravier and should be stated as a theorem.

THEOREM 2.8. (Gravier, [9]) Every non- k -choosable graph can be generated by the above three operations from any one non- k -choosable complete bipartite graph.

As Tuza states in his survey on list coloring[18], this result is quite fascinating. There is an increasing number of minimal complete bipartite graphs that are non- k -choosable as k gets large, but all of them are essentially equivalent from the generative point of view.

Using the probabilistic method, we can get an upper bound on the list chromatic number with respect to the chromatic number. The most commonly known bound is for a graph G of order n , $\chi_l(G) \leq \chi(G) \ln(n)$. The idea behind the proof is to construct a probability space based on partitions of the color classes and list assignments on the vertices of the graph. Show that a color is in one of the partitions with probability 1 divided by the number of partition sets. Through the wonders of the probabilistic method, we have that one such partition class exists, and it will give us a L -coloring using $\chi(G) \ln(n)$ colors.

The more interesting bounds were calculated by Noga Alon using the probabilistic method [1]. He answered a question that was originally proposed by Erdős, Rubin, and Taylor. Define K_{r*t} to be the complete r -partite graph with t vertices in each part. Alon showed in 1992 [1] that there exist constants c_1 and c_2 such that for every r and t

$$c_1 r \ln(t) \leq \chi_l(K_{r*t}) \leq c_2 r \ln(t).$$

The bound gives insight into the relationship between the chromatic number and list chromatic number. Moreover, if we take $t = |V(G)|$ and $r = \chi(G)$, then we get the bound

preceding Alon's result. This is particularly interesting in light of the previous section. We have seen that there exist complete bipartite graphs that are not k -choosable. Alon's result makes the upper bound tight.

2.5 The List-Chromatic Number of Planar Graphs

Revisiting a question aforementioned, do theorems that apply to $\chi(G)$ also apply to $\chi_l(G)$? Moreover, would the same technique used to prove an established fact concerning $\chi(G)$ also work to prove the analogous result for $\chi_l(G)$? With proving analogous theorems in list coloring, typically, the proofs are either the same, the result is true for list coloring, but the proof from vertex coloring fails, or the theorem is false. One of the best illustrations is the relationship between the chromatic number and list chromatic number of planar graphs. For showing the chromatic number and list chromatic number of planar graphs is less than or equal to six, the proofs are identical up to definition of terms. For less than or equal to five, the proof for the list chromatic number requires the construction of an ingenious induction hypothesis created by Thomassen; whereas, the proof for the chromatic number is much easier. Lastly, the famous 4-color theorem needs significant computer analysis to prove for the chromatic number; however, the result is false for list coloring!

THEOREM 2.9. If G is a planar graph, then $\chi(G) \leq 6$.

Proof. We proceed by induction. For the base case, the result is trivial for $|V(G)| \leq 6$. Assume the result holds for all planar graphs of degrees $1, 2, \dots, n-1$. Let G be a planar graph such that $|V(G)| = n$ and $|E(G)| = m$. By Euler's Formula for planar graphs, $m \leq 3n - 6$. This implies there exists a vertex $v \in V(G)$ such that $\deg(v) \leq 5$. Remove this vertex, and color the resulting graph, $G - v$, using the induction hypothesis. Add back v . Then v has at most five colored neighbors, so we have saved a color for v . \square

If we assign each vertex lists of size six, then v still has at most five colored neighbors. So we have saved a color from v 's list to color v . The same induction idea can be applied to show $\chi(G) \leq 5$ for planar graphs with a slight modification. In the proof, if we have a vertex of degree five, then we would fail to save a color by just deleting the vertex. Instead, we consider all neighbors adjacent to that vertex. It must be the case that two of the neighbors are not adjacent (if not, we would have a K_5 , which violates planarity). Merge these nonadjacent neighbors into a single vertex, and invoke the induction hypothesis on the resulting graph of $n - 2$ vertices. Since the neighbors are nonadjacent, when we un-merge, we keep the same color on both of the vertices. The neighbors of v use at most four colors, and we have saved a color to color v . Thus, $\chi(G) \leq 5$.

Before we show that $\chi_l(G) \leq 5$ for planar graphs, we first note that adding edges to a graph can only increase the list chromatic number. A *near triangulated graph* is an embedding of a graph in which every bounded face is a triangle. Once we have the following result for near triangulated graphs, then we can extend it to all planar graphs. To prove the theorem, we show a stronger result using Thomassen's induction hypothesis.

THEOREM 2.10. (Thomassen, [17]) Let G be a near triangulated graph, and let B be the outer cycle bounding the outer region. We make the following assumptions on the color sets $L(v), v \in V(G)$:

1. Two adjacent vertices x, y of B are already colored with different colors α and β .
2. $|L(v)| \geq 3$ for all other vertices v of B .
3. $|L(v)| \geq 5$ for all vertices v in the interior.

Then the coloring of x, y can be extended to a proper coloring of G by choosing colors from the lists. Moreover, $\chi_l(G) \leq 5$.

Proof. For the case of $|V| = 3$, the result is trivial since we have only one uncolored vertex, and since $|L(v)| \geq 3$, we have a color saved for this uncolored vertex. There are two cases to analyze, and we proceed using induction.

Case 1: We first suppose B has a chord (an edge not in B that joins two vertices $u, v \in B$, see Figure 2.2). The chord partitions the graph into two pieces. Consider the subgraph G_1 bounded by $B_1 \cup \{uv\}$, and let G_1 contain x, y, u , and v . G_1 is near triangulated, and by the induction hypothesis, has a coloring from its list. Let γ and δ be the colors assigned to vertices u and v in this coloring. Now consider the other subgraph G_2 . Since u and v are precolored in G_2 and since G_2 is a near triangulation, by our induction hypothesis, we can color G_2 from its list. Since G_1 and G_2 have colorings from their lists, and since the colorings agree on $V(G_1) \cap V(G_2) = \{u, v\}$, we have a coloring of G from its lists.

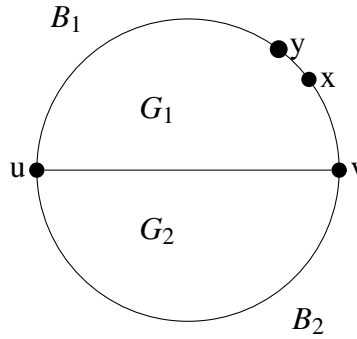


Figure 2.2: Case 1: Near-triangulated planar graph with a chord.

Case 2: Suppose that B does not have a chord. [See Figure 2.3] Let v_0 be a vertex on the other side of the α colored vertex x on B . Let $w, v_1, v_2, \dots, v_t, x$ be the neighbors of v_0 , where w and x are both on B . Construct the near triangulated graph $G' = G - v_0$ by deleting v_0 and all its incident edges. Then G' has a new outer boundary $B' = (B - v_0) \cup \{v_1, \dots, v_t\}$. Since $|L(v_0)| \geq 3$ by assumption, there exists two colors γ and δ in $L(v_0)$ that differ from α . For

$1 \leq i \leq t$, replace every color set $L(v_i)$ with $L(v_i) - \{\gamma, \delta\}$. Now, G' satisfies our induction hypothesis and can be colored from its lists. Add back v_0 . Then we have two possible colors γ and δ to color v_0 . Color v_0 with which ever color is not assigned to w . Thus, we have extended the coloring of G' to G .

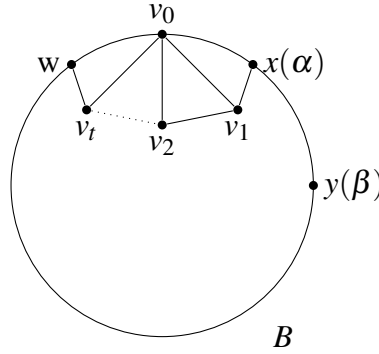


Figure 2.3: Case 2: Near-triangulated planar graph without a chord.

□

This leads us to one of the most famous problems in graph theory, and one of the problems that sparked a huge surge in research on coloring, the Four Color Theorem. It is well beyond the scope of this survey to attempt to provide any type of rigorous development of the proof of the Four Color Theorem, for the proof relies heavily on the use of computer computations. Kenneth Appel and Wolfgang Haken proved the result. They originally had 1,936 different possible configurations of counterexamples. With the proof of a highly challenging problem like the Four Color Theorem, scrutiny arose, and some mathematicians remain unconvinced as to the validity of the proof. More relevant to us is the fact that the theorem is false for list coloring!

A stronger conjecture was made concerning a relationship between the list chromatic number of a planar graph and the chromatic number. They believed the conjecture was false, and they challenged researchers to find a non 4-choosable planar graph. The conjecture was:

CONJECTURE 2.11. (Erdős, Rubin, and Taylor [8]) $\chi_l(G) \leq \chi(G) + 1$ for every planar graph G .

By Thomassen's proof, we have the case that $\chi(G) = 4$ implies $\chi_l(G) \leq 5$. An alternate definition for bipartite graphs is a graph is bipartite if and only if it is 2-colorable. Through a sophisticated use of orientations and digraphs, Alon and Tarsi showed that every planar, bipartite graph is 3-choosable [3]. This gives us the case that $\chi(G) = 2$ implies $\chi_l(G) \leq 3$. What if $\chi(G) = 3$? Does this $\chi_l(G) \leq 4$?

Shai Gutner constructed a graph that was later used by Margit Voigt to show that there exists a planar graph that is not 4-choosable [20]. In the original paper on characterizing two choosability of graphs, Erdős, Rubin, and Taylor conjectured that such a counterexample should exist. As with most constructions of counterexamples in list coloring, you build this basic, yet elegant, graph and take several copies of this graph by pasting certain vertices together.

Gutner constructed his graph with 130 vertices by using Figure 2.4.

We begin with 16 copies of graph H and identify all 16 copies of vertex α and identify all 16 copies of vertex β . The vertices labeled α and β are pasted together with sixteen copies of the above graph. The graph H has 10 vertices; we use 2 of the vertices sixteen times. Hence, the order of Gutner's graph is $16 * 10 - 16 * 2 + 2 = 130$. This graph is 3-colorable, and Voigt showed that by using these assigned lists and by choosing $\alpha \in \{5, 6, 7, 8\}$ and $\beta \in \{9, 10, 11, 12\}$ that the Gutner construction is not 4-choosable. The reader can easily check that with these lists a coloring is not possible. There are sixteen (α, β) combinations. We assign the list $\{5, 6, 7, 8\}$ to the vertex α and $\{9, 10, 11, 12\}$ to the vertex β . The α and β colors on the other vertices correspond to the sixteen pairs of (α, β) . Whichever α and β are chosen, we will have a subgraph, (one of the sixteen copies like the figure above), that will not yield a 4-list coloring. This means the graph on 130 vertices is also not 4-list

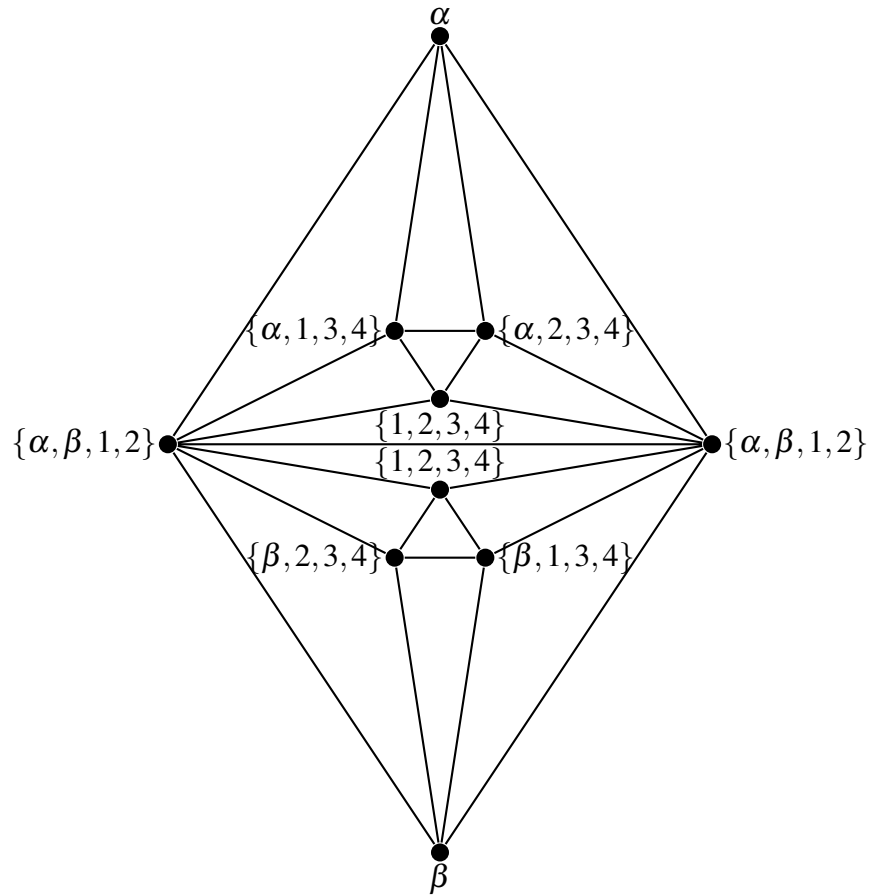


Figure 2.4: A "building block" H in a planar graph that is not 4-choosable.

colorable. Several other examples and constructions like the one above have been given to show the conjecture is false.

In list coloring, it would be convenient if the same proofs for theorems involving the chromatic number worked analogously. But in the case of planar graphs, we see that this is not true. We have to be more creative, but this is an attractive quality of list coloring. This leads us to our next question. Can we characterize choosability? In one of the earliest and most famous papers on list-coloring, Erdős, Rubin, and Taylor characterize 2-choosability.

2.6 2-Choosability of Graphs

If a graph G contains a vertex v of degree 1, then regardless of what color is used on its only neighbor u , we still have at least one color in v 's list to color v . Thus G is 2-choosable if and only if $G - v$ is 2-choosable. So in essence, we can prune away all vertices of degree 1. By doing so, we obtain a subgraph which Erdős, Rubin, and Taylor call the *core* of the graph. Erdős, Rubin, and Taylor showed that a graph is 2-choosable if and only if the core is 2-choosable. This successfully characterized 2-choosability of graphs. For coloring problems, we restrict ourselves to connected graphs, for if a graph is disconnected, then we simply color each component separately. Before proceeding with proving the 2-choosability characterization, we define a new type of graph.

A θ graph consists of two distinct vertices x and y together with three paths that are vertex disjoint except that each of the three paths meet at both x and y . We can easily name these θ graphs by specifying the path lengths. Here is an example of $\theta_{2,3,4}$.

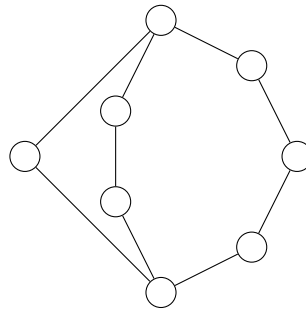


Figure 2.5: $\theta_{2,3,4}$ graph.

These graphs play an important role in our characterization, for they give us a potential instance of a possible core. Moreover, we note that the even cycles are a subgraph of these θ graphs. Our first goal is to show that $\theta_{2,2,2m}$ is 2-choosable for $m \geq 1$.

THEOREM 2.12. For $m \geq 1$, $\theta_{2,2,2m}$ is 2-choosable.

Proof. **Case 1:** All lists are the same, say $\{1, 2\}$.

Identify the two vertices in common with both paths and call them v_1 and v_{2m+1} . Label the vertices on the outside path of size $2m$ as v_2, v_3, \dots, v_{2m} clockwise from v_1 as seen in the below figure:

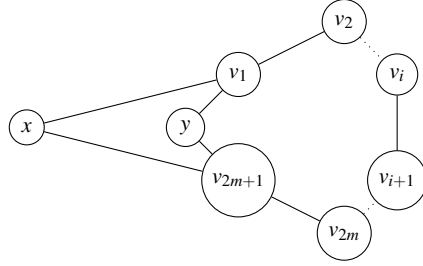


Figure 2.6: $\theta_{2,2,2m}$ graph for the proof.

For each vertex v_i with 1 if i is odd, and color v_i with 2 if i is even. So v_1 and v_{2m+1} receive color 1. The middle vertices, say x and y , in the two paths of length 2 are only adjacent to v_1 and v_{2m+1} , so color them both 2. Hence, we have a proper coloring.

Case 2: Not all lists are identical.

Keep the same labeling as above. Find a pair of adjacent vertices v_i and v_{i+1} such that $L(v_i) \neq L(v_{i+1})$. Tentatively consider the following coloring. Color v_i with a color from its list that is not in the list of v_{i+1} . Continue in this fashion counterclockwise until we color v_1 with color c_v . Now, we must look ahead to the list of v_{2m+1} . Let $L(v_{2m+1}) = \{c_1, c_2\}$. We must consider the colors of both x and y . If $L(x) \neq \{c_v, c_1\}$ and $L(y) \neq \{c_v, c_2\}$, then there is a color $c_{2m+1} \in L(v_{2m+1})$ such that $L(x) - \{c_v, c_{2m+1}\} \neq \emptyset$ and $L(y) - \{c_v, c_{2m+1}\} \neq \emptyset$. Choose those colors for x and y , and continue coloring counterclockwise around the outside path. When it is time to color v_{i+1} , we choose a color in $L(v_i)$ not in $L(v_{i+1})$. Hence, we can color v_{i+1} from its list. Therefore, we have a proper coloring. But if $L(x) = \{c_v, c_1\}$ and

$L(y) = \{c_v, c_2\}$, we must return to the original starting point where $L(v_i) \neq L(v_{i+1})$. Choose a color from $L(v_{i+1})$ that is not in $L(v_i)$, and color clockwise until we color v_{2m+1} with some color other than $c_v \in L(v_1)$. Choose colors c_v for vertices x and y . Then we have one color saved for v_1 . Color v_1 and continue coloring in clockwise fashion. By the same reasoning of choosing the different color in $L(v_{i+1})$, we have a proper coloring. \square

So $\theta_{2,2,2m}$ graphs are 2-choosable, and by consequence, even cycles are 2-choosable. Every 2-choosable graph has a core that is some subgraph of some $\theta_{2,2,2m}$ graph. Let $T = \{K_1, C_{2m+2}, \theta_{2,2,2m} | m \geq 1\}$. A graph is 2-choosable if and only if its core is one of the graphs in T . The proof of the characterization of 2-choosable graphs is by case analysis. We show by exhausting all possibilities that either the core of G is in T or the core of G contains a subgraph of one of the following five types:

1. An odd cycle.
2. Two vertex disjoint even cycles connected by a path.
3. Two even cycles having exactly one node in common.
4. $\theta_{a,b,c}$ where $a \neq 2$ and $b \neq 2$.
5. $\theta_{2,2,2,2m}$ where $m \geq 1$.

THEOREM 2.13. (A. L. Rubin, [8]) A graph G is 2-choosable if and only if the core of G belongs to T .

Proof. Assume G is not in T . If G contains an odd cycle, then we are finished. So we assume G is bipartite. Let C_1 be the shortest cycle. G must contain an edge not in C_1 , for if not, G would be an even cycle. If G contains a C_2 having at most one vertex in common with C_1 , then we have cases 2 or 3 above, and we are finished.

Let P_1 be a shortest path that is edge disjoint from C_1 and connects two distinct vertices of C_1 . If $C_1 \cup P_1$ is not in T , then the graph is in case 4, and we are finished. Now we must suppose $C_1 \cup P_1$ is in T . This implies that G is a $\theta_{2,2,2m}$, and C_1 must be a 4-cycle. So we let P_2 be the shortest path that is edge disjoint from $C_1 \cup P_1$ and connect two distinct vertices of $C_1 \cup P_1$. There are six cases to analyze. Name the vertices of C_1 as shown in the graph below of $C_1 \cup P_1$.

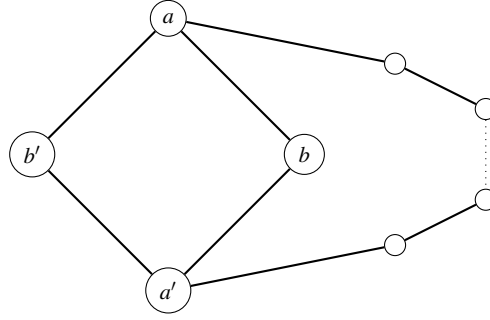


Figure 2.7: Figure for proof.

Case I: If the ends of P_2 are two interior vertices of P_1 , then we have a cycle disjoint from C_1 , and we are in case 2.

Case II: If the ends of P_2 are vertex a and an interior vertex of P_1 , then we have a cycle with exactly one vertex in common with C_1 , and we are in case 3.

Case III: If the ends of P_2 are vertex b and an interior vertex of P_1 , then we have a path from a to b that is edge disjoint from C_1 . Hence, we are in case 4.

Case IV: If the ends of P_2 are both a and b , then we are in case 4 again (similar to case III).

Case V: If the ends of P_2 are a and a' and if P_1 is length 2, then we are in case 5. If P_1 is of length greater than 2, then we are in case 4.

Case VI: If the ends of P_2 are b and b' , then by removing an edge from C_1 , we have a θ graph, and we are in case 4.

So if G is not in T , then G contains one of the five types. We need only show that the

five types are not 2-choosable. We first note that type 1 is not 2-colorable. We proceed with types 2, 3, 4, and 5 by building a reduction argument that allows us to reduce each type down to a simpler graph. Then we construct lists of size two that do not permit a proper coloring of these simpler reductions.

For the reduction, choose a vertex v , and contract all edges incident to v . Since the graph is bipartite, we will have no loops, and multiple edges can be made single. If the resulting graph G' is not 2-choosable, then G is not 2-choosable.

So suppose G' is not 2-choosable. Assign the list $\{x, y\}$ to v . Uncontract all edges incident to v and assign each of the new vertices the same list as v . If we choose color x for vertex v , then we must choose y for all vertices adjacent to v . So a choice for G would have worked for G' . After a repeated use of the reduction process for the types 2, 3, 4, and 5, we need only construct list assignments to demonstrate that the following 4 graphs are not 2-choosable.

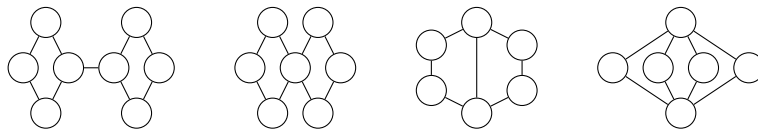


Figure 2.8: The graph reductions for types 2, 3, 4, and 5 respectively.

We assign the vertices in each graph lists of size 2 that do not permit a proper coloring. See Figures 2.9 - Figure 2.11 on the next page. □

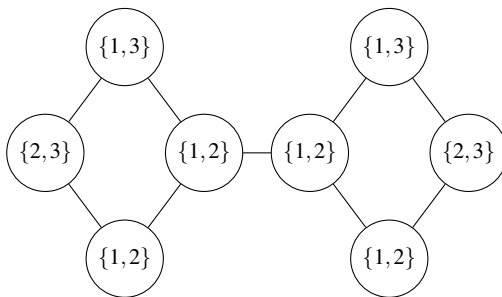


Figure 2.9: Type 2.

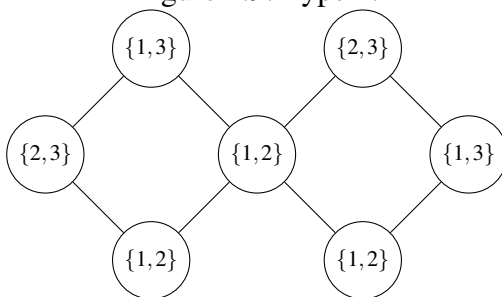


Figure 2.10: Type 3.

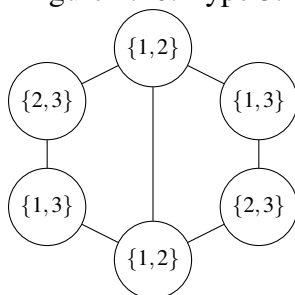


Figure 2.11: Type 4.

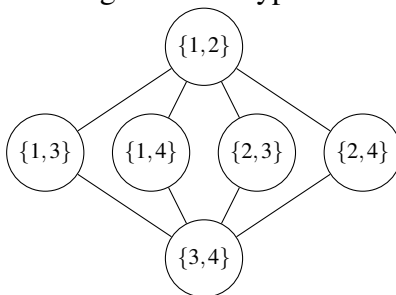


Figure 2.12: Type 5.

The Sum Choice Number of a Graph

3.1 Definition and First Results

Most of the research on the sum chromatic number has been conducted by Garth Isaak and Brian Heinold, who was a student under Garth Isaak. For a survey of sum chromatic number, the reader should refer to [12]. Daniel Cranston answered several questions on the sum chromatic number in [6]. Moreover, most of my initial thesis research was conducted in the sum chromatic number. Before proceeding, we define a function f to specify list sizes for each vertex. A graph is f -choosable if it can be properly colored from the assigned lists. Define $\sum_{v \in V(G)} f(v)$ to be the *size of f* . The *sum-chromatic-number*, $\chi_{sc}(G)$, is the minimum size of a choosable f . We also refer to the sum chromatic number as the sum choice number.

Two papers by Garth Isaak [13,14] deal with the sum choice number of block graphs and $2 \times n$ arrays. The greedy algorithm is still applied to problems involving the sum choice number. An important first result using the greedy algorithm is proved by Isaak in [14]. He shows that for any graph G , $\chi_{sc}(G) \leq |V(G)| + |E(G)|$. Let $b(G) = |V(G)| + |E(G)|$, where b stands for bound. Graphs for which $\chi_{sc}(G) = b(G)$ are called *sum choice greedy*. One significant open problem in this area is to characterize which graphs are sum choice greedy. Given what we do know about χ_{sc} , it seems like this quest is futile. Instead of formalizing a characterization of sum choice greedy graphs, we try to find sufficient conditions for sum choice greedy and non sum choice greedy graphs.

Cliques or graphs with a high clique number force a higher chromatic number. Recall that the block of a graph is a maximal 2-connected subgraph. If the graph is 2-connected,

then the whole graph is considered a block. A *block graph* is a graph such that every block is a clique. In [14], Isaak shows that block graphs are sum choice greedy. This also characterizes the sum choice number for trees because a tree is a block graph.

Does this equality hold for other classes of graphs? More research needs to be conducted on specific classes of graphs. In his dissertation, Heinold discovered several graphs that are sum choice greedy. One interesting result relies on the 2-choosability characterization of Erdos, Rubin, and Taylor. Theta graphs are sum choice greedy with the exception of $\theta_{2,2,2m}$ graphs. The easiest example of a $\theta_{2,2,2m}$ graph that is not sum choice greedy is $K_{2,3}$. Note that $b(K_{2,3}) = 11$. However, $K_{2,3}$ is 2-choosable. Since $|V(K_{2,3})| = 5$, this implies we have used a sum of only 10, which is better than 11. A more interesting and technically advanced example is that the Petersen graph is sum choice greedy, which Heinold showed in his dissertation [11]. The reason that this example is of particular importance to us is that it collects and applies several known and necessary techniques used in sum choice results, and it will be the climax of this exposition.

Isaak also calculated the sum choice number of $2 \times n$ arrays [13] which is $n^2 + \lceil \frac{5n}{3} \rceil$. Isaak also gives a detailed case analysis showing that the 2×3 array has sum choice number 14. Many interesting questions have been initiated based on this result. It is not known what the sum choice number of the 3×3 grid is; although, it is known that it must be at least 24 by an analysis of Isaak. In a paper by Berliner, Bostelmann, Brualdi, and Deaett, they showed [5] that $\chi_{sc}(K_{2,q}) = 2q + 1 + \lfloor \sqrt{4q + 1} \rfloor$. Heinold later showed [11] that $\chi_{sc}(K_{3,q}) = 2q + 1 + \lfloor \sqrt{12q + 1} \rfloor$.

Coloring under sum choice restraints can be very difficult. We must develop strategies for saving colors from lists or deleting vertices that have special list sizes. The latter strategy is analogous to the proof of the 5-choosability of planar graphs. We can remove a vertex and save a color by coloring all neighbors. This is extremely helpful when sum-list-coloring the Petersen graph. The other difficulty in sum-list-coloring is that we can permute list sizes

with the vertices, and this can lead to tedious case analysis. For the scope of this exposition and introduction to sum choice number, coloring the Petersen graph is a formidable task and requires several lemmas. If anyone is to pursue any work in sum list coloring, the techniques and lemmas used to prove that $\chi_{sc}(\text{Petersen}) = 25$ are fundamental.

3.2 Two Important Lemmas

We need two lemmas proved by Isaak in [14]. They are used in several induction proofs in Isaak's papers. We need to invoke a special case of the following lemmas for our proofs later.

LEMMA 3.1. Assume that vertex list sizes f are given for an arbitrary graph G and for some k clique B in G , we have $f(v) \leq k$ for every $v \in B$. Assume also that all vertices in B have the same neighborhood $N(B)$ outside B . List the sizes assigned to the vertices of B in non-decreasing order $t_1 \leq t_2 \leq \dots \leq t_k \leq k$. Then:

1. If $t_i < i$ for some $i \in \{1, 2, \dots, k\}$, then G is not f -choosable.
2. If $t_i \geq i$ for all $i \in \{1, 2, \dots, k\}$, then let G' be the graph induced by $V(G) - B$ and assign list sizes to G' as follows: $f'(w) = f(w)$ for $w \notin N(B)$ and $f'(w) = f(w) - k$ for $w \in N(B)$. Then G is f -choosable if and only if G' is f' -choosable.

Proof. (1.) Label the vertices of B as v_1, v_2, \dots, v_k so that $f(v_i) = t_i$. Assigning initial lists to v_1, v_2, \dots, v_i allows fewer than i colors for this i clique. Hence, there is no proper coloring in this case. Thus, G is not f -choosable.

(2.) Assume that G is f -choosable. Given lists L' with sizes $f'(v)$ for $v \in V(G')$, relabel so that the colors are all distinct from $\{1, 2, \dots, k\}$. Form lists L with sizes $f(v)$ for $v \in V(G)$ by taking $C(v)$ initial size $f(v)$ for $v \in B$, $C(w) = C'(w)$ for $w \in V(G') - N(B)$ and $C(u) = C'(u) \cup \{1, 2, \dots, k\}$ for $u \in N(B)$. Since G is f -choosable, there is a proper L -coloring. In

any any such coloring, colors 1, 2, ..., k must be used on B since B is a clique and since $t_i \geq i$ for all i along with $t_k \leq k$ imply $t_k = k$. Then these colors are not used on $N(B)$. So the coloring restricted to G' is a proper L' -coloring after we relabel with the original colors.

Conversely, assume that G' is f -choosable. Given lists with sizes $f(v)$ for $v \in V(G)$, color the vertices using these lists. Greedy coloring using the ordering given by non-decreasing list sizes in B will yield a coloring because of the condition $t_i \geq i$. Let A be the set of colors used on B . For $v \in V(G')$ form lists with $C'(v) = C(v) - A$. Then the sizes of the lists are at least those given by f' . Hence, the remaining vertices of G' are colorable. Since the colors in C' are distinct from A , the coloring G' along with that of B gives a proper list coloring. \square

We need one more lemma from Isaak's paper [14]. The proof technique is similar to that for Lemma 3.1, so we omit the proof.

LEMMA 3.2. Assume vertex list sizes f are given for G . Let W be a subset of $V(G)$ such that all vertices have the same neighborhood $N(W)$ outside W and let G' be the graph induced by $V(G) - W$ and G'' the graph induced by W . Define f' on G' to be f restricted to $V(G) - W$ and define f'' on W by $f''(w) = f(w) - |N(W)|$ for all $w \in W$. If G'' is f'' -choosable, then G is f -choosable if and only if G' is f' -choosable.

With these two results, we can develop techniques to simplify our case analysis for sum-list-coloring graphs. The ultimate goal is to color the Petersen graph. The purpose of the proof is to show that these sum coloring problems and characterizations can be difficult.

3.3 Gathering the Tools

Given a pair (G, f) , a vertex is *forced* by an f -assignment C if it receives the same color in any proper C -coloring of G . For any vertex $v \in V(G)$, we define the size function f^v

on $G - v$ by $f^v(w) = f(w) - 1$, if w is adjacent to v , and $f^v(w) = f(w)$ otherwise. For simplicity, we define two new parameters:

$$\rho(G) = \min\{\chi_{sc}(G - v) + \deg(v) + 1 : v \in V(G)\},$$

$$\tau(G) = \min\{\text{size}(f) : f \text{ is choosable, and } 2 \leq f(v) \leq \deg(v) \text{ for all } v \in V(G)\}.$$

The following lemma is a special case of lemmas 3.1 and 3.2 in the previous section, and it gives us a technique for proving sum choice problems.

- LEMMA 3.3. 1. If $f(v) = 1$ for some vertex $v \in V(G)$, then (G, f) is choosable if and only if $(G - v, f^v)$ is choosable.
2. If $f(v) > \deg(v)$ for some vertex v , then (G, f) is choosable if and only if $(G - v, f_{G-v})$ is choosable.

Lemma 3.3 is important because it reduces our case analysis. If we have a vertex with list size 1 or size greater than its degree, the case can be tedious and pestering to prove. The following lemma allows us to preclude these cases.

LEMMA 3.4. For any graph G , $\chi_{sc}(G) = \min\{\rho(G), \tau(G)\}$. In particular, if $G - v$ is sum choice greedy for every $v \in V(G)$, then $\chi_{sc}(G) = \min\{b(G), \tau(G)\}$.

We will use lemma 3.4 for sum-list-coloring theta graphs and the Petersen graph. Before proceeding to some of the more interesting and technical proofs for the sum choice number, it is beneficial to formally state the major results from [14].

LEMMA 3.5. If G is a graph decomposable into blocks G_1, G_2, \dots, G_k , then

$$\chi_{sc}(G) = \sum_{j=1}^k \chi_{sc}(G_j) - k + 1.$$

Summarizing the above three results, we have a result that allows us to preclude the cases where a list size is $f(v) = 1$ or $f(v) > \deg(v)$. For simplicity, call these assignments *simple size functions*. Moreover, if the blocks of a graph are all sum choice greedy, then the

graph itself is sum choice greedy (i.e. $\chi_{sc}(G) = |V(G)| + |E(G)|$). In light of Lemma 3.5, some immediate observations are that cycles, complete graphs, and trees are sum choice greedy.

3.4 A Warm Up Before Coloring the Petersen Graph

Recall from Section 2.6 the theta graph. θ_{k_1, k_2, k_3} is a simple graph consisting of two vertices connected by three internally vertex disjoint paths with $0 \leq k_1 \leq k_2 \leq k_3$ where k_i denotes the number of edges in each path. For theta graphs, $b(\theta_{k_1, k_2, k_3}) = 2(k_1 + k_2 + k_3) - 1$. Then:

THEOREM 3.6. If $G = \theta_{k_1, k_2, k_3}$, then

$$\chi_{sc}(G) == \begin{cases} b(G) - 1 & : \text{if } k_1 = k_2 = 2 \text{ and } k_3 \text{ is even,} \\ b(G) & : \text{otherwise} \end{cases}$$

Proof. If we remove a vertex from a theta graph, then we have a graph that is either a tree or a cycle with pendant edges. Both of these are sum choice greedy. So by Lemma 3.4, we need to determine $\tau(\theta_{k_1, k_2, k_3})$. If f is a non-simple size function with $\text{size}(f) = b(G) - 1 = 2(k_1 + k_2 + k_3) - 2$, then $f(v) = 2$ for all $v \in V(G)$, since the vertex set has size $k_1 + k_2 + k_3 - 1$. But in Section 2.6, we discovered that the only theta graphs that are 2 choosable have $k_1 = k_2 = 2$ and k_3 even. \square

As a culmination of all of the material on the sum choice number, we now look at a proof that $\chi_{sc}(\text{Petersen}) = 25$. Here is our strategy. By Lemma 3.4, to show that a graph G is sum choice greedy, we need only show that $G - v$ is also sum choice greedy with the condition that we cannot improve the bound when there is no non-simple size function of size $b(G) - 1$. Now, we can recursively use this process until we find a graph that we know

to be sum choice greedy, and at each iteration of the reduction, we need only show that there are no choosable non-simple size functions of size $b(G) - 1$.

3.5 Sum-Coloring the Petersen Graph

THEOREM 3.7. (Heinold, [11]) The Petersen graph is sum choice greedy; that is, the sum choice number is 25.

Proof. Let P be the Petersen graph, Q be the graph obtained by removing one vertex of the Petersen graph, and R be the graph obtained by removing a vertex of degree 2 from Q . Figure 3.1 shows P , Q , and R . Note, $b(P) = 25$, $b(Q) = 21$, and $b(R) = 18$.

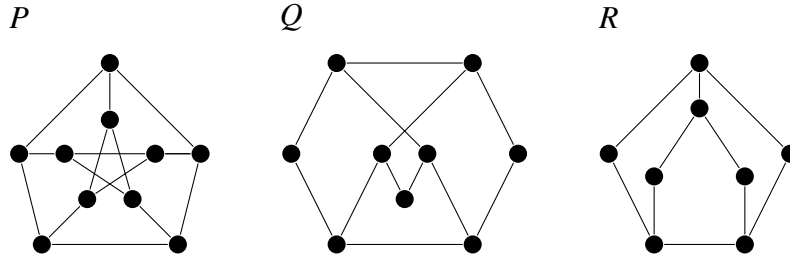


Figure 3.1: Graphs of P , Q , and R respectively.

By Lemma 3.4, if we attach a leaf to a sum choice greedy graph, then the resulting graph is also sum choice greedy. For any vertex $v \in V(R)$, $R - v$ is either a theta graph or a cycle with pendant edges. This implies R is sum choice greedy by Lemma 3.5 and Theorem 3.6. For any vertex v of degree 3 in $V(Q)$, $Q - v$ is a sum choice greedy theta graph with pendant edges. Thus, it remains to examine non-simple size functions of size one less than the $b(H)$ for each choice of $H \in \{P, Q, R\}$.

The only non-simple size functions of size 17 on R assign list size 2 to all but one vertex. Hence, there is a C_5 with lists of size 2 only. But odd cycles are not 2-choosable.

The only non-simple size functions of size 20 on Q assign list size 2 to all vertices except

two vertices v and w , both of degree 3. There are 5-cycles that avoid any pair of adjacent vertices in Q . So if v and w are adjacent, then we have a 5-cycle with list sizes all 2, which is not 2-choosable. If v and w are not adjacent, then they must be a distance of 2 apart. Let x denote their common neighbor. Then there exists two 5-cycles C_1 and C_2 with v in C_1 but not in C_2 , w in C_2 but not in C_1 , and x in neither. We construct a counterexample. Let f be a list assignment L of size 20. Let $L(v) = \{1, 2, 3\}$, $L(w) = \{1, 2, 4\}$, $L(x) = \{3, 4\}$, and let every other vertex have list $\{1, 2\}$. These lists will force color 3 on v and color 4 on w . Hence, x will be uncolored. Thus, there is no proper coloring from these lists.

We need only consider non-simple size functions of size 24 on P . Any size function assigns lists of size 3 to four vertices and lists of size 2 to all others. Several cases will arise. By symmetry, we can reduce these cases to 6 possible configurations of P with lists. We consider a subgraph argument. Let H be the subgraph induced by the vertices assigned list size 3. The cases will be considered based on the possibilities of H .

If H is a P_4 , then there exists a 5-cycle whose list sizes are all 2, which is not 2-choosable. The other possibilities for H are (a.) $K_{1,3}$, (b.) $P_3 + K_1$, (c.) $K_2 + K_2$, (d.) $K_2 + K_1 + K_1$, or (e.) 4 isolated vertices. See Figures 3.2, 3.3, and 3.4 for the list assignments. The bold vertices represent cases (a) - (e), for they are the vertices with lists of size three. The given lists will show that each configuration has a size function of size 24 that will not yield a proper coloring of P . The reader should note that by symmetry, these are the only possible configurations.

Use Figure 3.2 for (a) and (b). For (a), the vertices with lists 123, 124, and 125 are each contained in different 5-cycles whose other vertices have lists 12. So, colors 3, 4, and 5, respectively, are forced on those vertices. The top vertex cannot be colored. For (b), the inside star is a 5-cycle. This forces color 3 on the isolated vertex of H . Then we must color the vertex adjacent to the isolated vertex on the outside 5-cycle with color 4. By doing so,

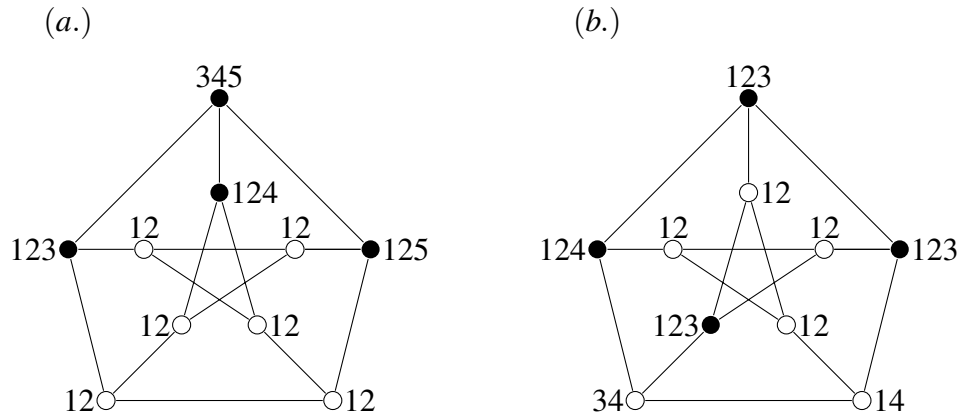


Figure 3.2: Cases (a.) and (b.) with lists.

the choices force the top vertex to be uncolored. So (b) is un-colorable from the remaining lists.

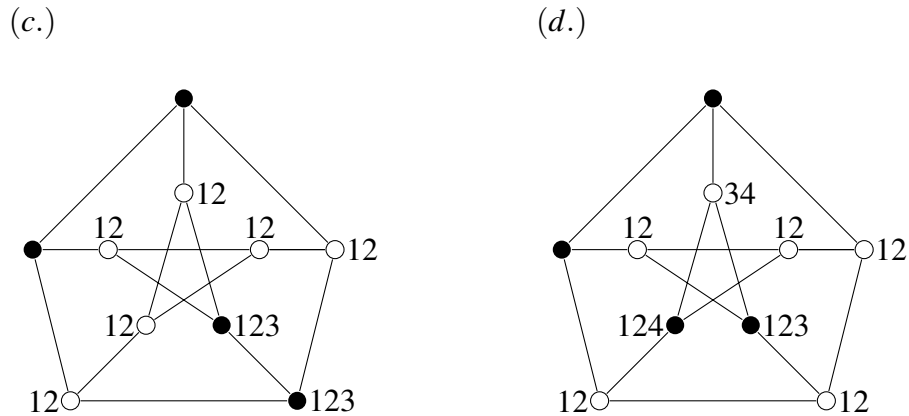


Figure 3.3: Cases (c.) and (d.) with lists.

Use Figure 3.3 for (c) and (d). For (c), color 3 is forced on both of the vertices with list 123 as both are on 5-cycles whose remaining vertices have lists 12. As these two vertices are adjacent, one of them will not be allowed to receive color 3. For (d), colors 3 and 4 are forced on the vertices with lists 123 and 124, respectively, for both of these vertices are contained on 5-cycles whose other vertices have lists 12. The vertex with list 34 is adjacent

to both of these vertices, and it cannot be colored from its list.

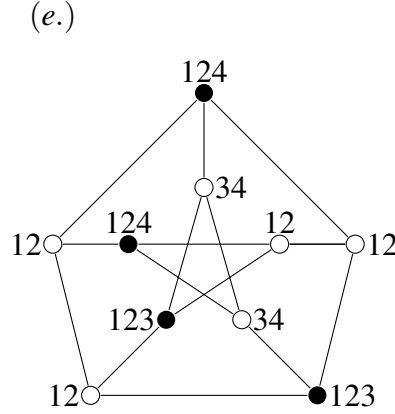


Figure 3.4: Case (e.) with lists.

Use Figure 3.4 for (e). We must consider two cases. Suppose the bottom right vertex with list 123 receives color 3. Then color 4 must be used on the inside vertex adjacent to it with list 34. This implies color 3 must be used on the other inside vertex with list 34. Hence, the inside vertices with list 123 and 124 cannot be colored with colors 3 and 4, respectively. But a proper coloring must use color 3 or color 4 on one of these vertices. For if not, the 5-cycle containing those two vertices and three vertices with list 12 would not be choosable. Thus, no proper coloring exists when we choose color 3 on the bottom right vertex with list 123. Now for the second case, we keep the same lists, but by the above analysis, we regard the bottom right vertex as only allowing 1 or 2 to be selected. By assuming so, colors 3 and 4 are forced on the leftmost vertex with list 123 and the topmost vertex with list 124, respectively, since both of these vertices are contained in 5-cycles whose other vertices have list 12. These two vertices have a common neighbor with list 34. This vertex cannot be colored. \square

Even if the reader has no interest in pursuing research in sum coloring graphs, one

should be able to appreciate the beauty of the technique in showing the Petersen graph is sum choice greedy. Without the lemmas to preclude the simple size functions, this problem would be more tedious. The reduction argument of finding non 2-choosable 5 cycles is a common technique in list coloring. If we can find a subgraph that is not k -choosable, then the entire graph is not k -choosable. Since with list coloring a coloring must exist no matter what lists are given, finding a desired counterexample can be rather difficult. But the central idea in this exposition is that no matter what choices we make, we want to force one vertex to always have an empty list after we color all of its neighbors.

Concluding Remarks and Future Research

This survey is, by no means, a complete collection of all known facts in list coloring. We chose to investigate the sum choice number as a specialization of list coloring. Other areas include edge list coloring, fractional coloring, and several other specializations of list coloring. We have been choosing one color for each vertex. One of the more challenging and rather fun problems of list coloring is trying pick two, three, or, in general, k colors for each vertex from the list while preserving a proper coloring? Daniel Cranston investigated this problem in the context of the sum choice number in [6]. Cranston showed that under certain sum sizes, you can pick two colors for each vertex for paths and cycles. This leads to a lot of case analysis, and the original pursuit of the research was to generalize these ideas and determine under what conditions we can pick k colors for each vertex from its lists for paths and cycles.

In light of what has been explored in the previous chapter, there are still several unanswered questions involving list coloring and the sum choice number. The reader is encouraged to check current research archives for an updated list of results since graph theory is seemingly always blossoming with new results. Some questions to ponder are what about sum choice numbers of products, unions, special graphs like snarks, or other known classes of graphs? Ideally, we would like to characterize all graphs under the sum choice greedy constraint, but this is seemingly a hopeless endeavor. However, that does not imply that it will not be fun or rewarding to try to do so!

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